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THE JACOBIAN PROBLEM AS A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

Let $P = x^n + P_{n-1}(y)x^{n-1} + \cdots + P_0(y), Q = x^m + Q_{m-2}(y)x^{m-2} + \cdots + Q_0(y)$ belong to $K[x,y]$, where K is a field of characteristic zero. The main result of this paper is the following: Assume that $P_xQ_y - P_yQ_x = 1$. Then:*

(i) $K[Q_{m-2}(y), \ldots, Q_0(y)] = K[y]$,
(ii) $K[P,Q] = K[x,y]$ if $Q = x^m + Q_k(y)x^k + Q_r(y)x^r$.

Introduction

Let K be a field of characteristic zero and let $P, Q \in K[x,y]$ satisfy the Jacobian identity: $P_xQ_y - P_yQ_x = 1$. Such a pair of polynomials is called a Jacobian pair.

If $P = x^n + P_{n-1}(y)x^{n-1} + \cdots + P_0(y), Q = x^m + Q_{m-2}(y)x^{m-2} + \cdots + Q_0(y)$, and $m \geq 2$, then we have a reduced Jacobian pair. It is easy to see that every Jacobian pair can be transformed into a reduced one. In this paper we will be interested in special properties of reduced Jacobian pairs. The famous Jacobian Conjecture states that every Jacobian pair generates $K[x,y]$. To the best of my knowledge, it is still unproven.

Perhaps a brief historical note is in order. Let $d_1 = \deg P, d_2 = \deg Q$. There are proofs of the Jacobian Conjecture for special values of d_1 and d_2. Magnus ([6], 1955) showed that the conjecture is true if d_1 or d_2 is prime. Then Nakai and Baba ([9], 1977) proved the conjecture if d_1 or d_2 is 4 or $d_1 > d_2, d_1 = 2p$ and

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$p > 2$ is prime. This result was generalized by Appelgate and Onishi ([2], 1985): they proved the conjecture for the case when d_1 or d_2 has at most two prime factors. This result was reproved by Nagata ([8], 1988). See also Nowicki ([10], 1988). In 1983 Moh ([7]) showed, using a computer search, that the conjecture is true if both d_1 and $d_2 \leq 100$. An excellent review of the problems related to the Jacobian Conjecture and of several faulty proofs can be found in [3].

In this paper we develop a technique which enables us to prove the Jacobian Conjecture in some partial cases and gives a new and promising perspective on the general case. The technique is based on so-called Toeplitz sequences. These sequences are defined and their properties are established in Section 1, which is purely technical and should be mainly used as a reference section. The first non-trivial result is obtained in Theorem 2.2: If P, Q is a reduced Jacobian pair, then $K[Q_{m-2}(y), \cdots, Q_0(y)] = K[y]$. So every reduced Jacobian pair determines a certain embedding of a line into K^{m-1}. Special properties of this embedding are studied in Section 2 and Section 3. Proposition 2.4 establishes necessary and sufficient conditions for $Q = x^m + Q_{m-2}(y)x^{m-2} + \cdots + Q_0(y)$ to have a Jacobian mate of x-degree n. (A result in this direction was obtained in [4]. Both the result and the technique used are quite different from ours). In Section 3 the so-called fundamental system for P, Q is introduced. This system of polynomial equations in $Q_{m-2}(y), \ldots, Q_0(y)$ has very special properties and contains crucial information about the embedding of a line into K^{m-1} determined by P, Q. The main result of Section 3 is that the Jacobian determinant of the fundamental system is a non-zero constant. This fact is used in Section 4 to prove a special case of the Jacobian Conjecture, which is, I believe, quite a new result (Theorem 4.1). The proof is based on the results of Section 3 and on the Abhyankar–Moh Theorem. Since there exists a result similar to the Abhyankar–Moh Theorem for higher-dimensional spaces, it looks very promising to try methods similar to the ones used in Section 4 in the general case.

1. Toeplitz sequences

Throughout this paper K will denote a fixed algebraically closed field of characteristic zero.

Let k be an integer and let r be a non-negative integer. Let

$$F = \{\ldots, F_i, F_{i-1}, \ldots, F_k\}$$

be a sequence with the following properties:

(i) The terms F_i belong to $K[z_r, \cdots z_0]$.

(ii) The sequence F is infinite from the left and terminates at the index k.

Let $S_{k,r}$ denote the K-linear space formed by all such sequences.

Definition 1.1: A sequence $F \in S_{k,r}$ is a **Toeplitz sequence** if the following conditions hold for all $i \geq k$:

$$\text{(i)} \qquad \frac{\partial F_{i+1}}{\partial z_{j+1}} = \frac{\partial F_i}{\partial z_j} \quad \text{for } 0 \leq j < r,$$

$$\text{(ii)} \qquad \frac{\partial F_i}{\partial z_r} = \frac{i+2}{r+2} F_{i+2} - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial F_{i+2}}{\partial z_j}.$$

All Toeplitz sequences belonging to $S_{k,r}$ form a K-linear subspace $T_{k,r}$ of $S_{k,r}$. We will primarily be interested in a special class of Toeplitz sequences:

Definition 1.2: A Toeplitz sequence $F = \{\ldots, F_n, \ldots, F_k\}$ is **of height** n if $F_n \neq 0$ and $F_i = 0$ for $i > n$.

All Toeplitz sequences of height at most n, belonging to $T_{k,r}$ form a K-linear subspace $T^n_{k,r}$ of $T_{k,r}$. Elements of $T^n_{k,r}$ will be written in the form $\{F_n, \ldots, F_k\}$ with the understanding that there are infinitely many zeroes to the left of F_n.

For every pair of integers $n > k$ we define the **truncation map** $\tau^n_{k,r}\colon T^n_{k,r} \to T^n_{k+1,r}$ in the obvious way:

$$\tau^n_{k,r}(\{F_n, \ldots, F_{k+1}, F_k\}) = \{F_n, \ldots, F_{k+1}\}.$$

LEMMA 1.3: $\operatorname{Ker} \tau^n_{k,r} \simeq K$.

Proof: Ker $\tau^n_{k,r}$ consists of all elements of $T^n_{k,r}$ of the form $\{0, \ldots, 0, F_k\}$. These elements are Toeplitz sequences. Therefore:

$$\text{(i)} \qquad \frac{\partial F_k}{\partial z_j} = \frac{\partial F_{k+1}}{\partial z_{j+1}} = 0 \qquad \text{for } j < r.$$

$$\text{(ii)} \qquad \frac{\partial F_k}{\partial z_r} = \frac{k+2}{r+2} F_{k+2} - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial F_{k+2}}{\partial z_j} = 0.$$

Thus $F_k \in K$, which concludes the proof.

PROPOSITION 1.4: *$\tau^n_{k,r}$ is surjective.*

Proof: We have to prove that for every Toeplitz sequence $\{F_n, \ldots, F_{k+1}\}$ there exists a polynomial $F_k \in K[z_r, \ldots, z_0]$ such that the sequence $\{F_n, \ldots, F_{k+1}, F_k\}$ is Toeplitz. Set:

$$\begin{cases} f_j = \frac{\partial F_{k+1}}{\partial z_{j+1}} & \text{for} \quad 0 \le j < r, \\ f_r = \frac{k+2}{r+2} F_{k+2} & - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial F_{k+2}}{\partial z_j}. \end{cases}$$

Now we have to prove that $\partial f_j / \partial z_i = \partial f_i / \partial z_j$. To do this, we have to consider the following two cases:

CASE A: $i < r,\ j < r$.
In this case

$$\frac{\partial f_j}{\partial z_i} = \frac{\partial^2 F_{k+1}}{\partial z_i \partial z_{j+1}} = \frac{\partial \left(\frac{\partial F_{k+1}}{\partial z_i} \right)}{\partial z_{j+1}} = \frac{\partial \left(\frac{\partial F_{k+2}}{\partial z_{i+1}} \right)}{\partial z_{j+1}} = \frac{\partial^2 F_{k+2}}{\partial z_{i+1} \partial z_{j+1}}$$

and

$$\frac{\partial f_i}{\partial z_j} = \frac{\partial^2 F_{k+1}}{\partial z_{i+1} \partial z_j} = \frac{\partial \left(\frac{\partial F_{k+1}}{\partial z_j} \right)}{\partial z_{i+1}} = \frac{\partial \left(\frac{\partial F_{k+2}}{\partial z_{j+1}} \right)}{\partial z_{i+1}} = \frac{\partial^2 F_{k+2}}{\partial z_{j+1} \partial z_{i+1}}.$$

So in this case $\partial f_i / \partial z_j = \partial f_j / \partial z_i$.

CASE B: $i < r,\ j = r$.
In this case

$$\begin{aligned} \frac{\partial f_r}{\partial z_i} &= \frac{k+2}{r+2} \frac{\partial F_{k+2}}{\partial z_i} - \frac{i}{r+2} \frac{\partial F_{k+2}}{\partial z_i} - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial^2 F_{k+2}}{\partial z_i \partial z_j} \\ &= \frac{k+2-i}{r+2} \frac{\partial F_{k+2}}{\partial z_i} - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial^2 F_{k+2}}{\partial z_i \partial z_j}. \end{aligned}$$

On the other hand

$$\begin{aligned} \frac{\partial f_i}{\partial z_r} &= \frac{\partial^2 F_{k+1}}{\partial z_r \partial z_{i+1}} = \frac{\partial \left(\frac{\partial F_{k+1}}{\partial z_r} \right)}{\partial z_{i+1}} = \frac{\partial \left(\frac{k+3}{r+2} F_{k+3} - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial F_{k+3}}{\partial z_j} \right)}{\partial z_{i+1}} \\ &= \frac{k+3}{r+2} \frac{\partial F_{k+3}}{\partial z_{i+1}} - \frac{i+1}{r+2} \frac{\partial F_{k+3}}{\partial z_{i+1}} - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial^2 F_{k+3}}{\partial z_j \partial z_{i+1}} \\ &= \frac{k+2-i}{r+2} \frac{\partial F_{k+2}}{\partial z_i} - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial^2 F_{k+2}}{\partial z_j \partial z_i} = \frac{\partial f_r}{\partial z_i}. \end{aligned}$$

This concludes the proof of the fact that $\partial f_i/\partial z_j = \partial f_j/\partial z_i$ for all permissible i, j. There exists, therefore, a polynomial $F_k \in K[z_r, \ldots, z_0]$ so that $\partial F_k/\partial z_i = f_i$ for $0 \le i \le r$. Recalling the definition of the f_i's, we see that $\{F_n, \ldots, F_{k+1}, F_k\}$ is a Toeplitz sequence. This concludes the proof. ∎

PROPOSITION 1.5: $\dim T^n_{k,r} = n - k + 1$.

Proof: The proof is by induction on $n - k$:

(i) $n - k = 0$.

In this case $\dim T^n_{n,r} = 1$ since, obviously, $\dim T^n_{n,r} \simeq K$.

(ii) $n - k > 0$ and $\dim T^n_{k+1,r} = n - k$.

In this case we have a sequence of K-linear spaces:

$$0 \to \operatorname{Ker} \tau^n_{k,r} \to T^n_{k,r} \xrightarrow{\tau^n_{k,r}} T^n_{k+1,r} \to 0.$$

This sequence is exact since $\tau^n_{k,r}$ is surjective by Proposition 1.4. Then $\dim T^n_{k,r} = n - k + 1$ since $\dim \operatorname{Ker} \tau^n_{k,r} = 1$ by Lemma 1.3 and $\dim^n_{k+1,r} = n - k$ by our induction hypothesis. This concludes the proof. ∎

Now we are able to construct a convenient basis for $T^n_{k,r}$. For $s \le n$ set

$$G_{s,n} = \sum_{2i_r + \cdots + (r+2)i_0 = n-s} \frac{\prod_{t=0}^{i_r + \cdots + i_0 - 1} [n - t(r+2)]}{i_r! \cdots i_0! (r+2)^{i_r + \cdots + i_0}} z_r^{i_r} \cdots z_0^{i_0}.$$

Remark: Note that $G_{n,n} = 1, G_{n-1,n} = 0$.

LEMMA 1.6:

$$\sum_{j=0}^{r} (r + 2 - j) z_j \frac{\partial G_{s,n}}{\partial z_j} = (n - s) G_{s,n}.$$

Proof: Straightforward.

LEMMA 1.7:

$$\frac{\partial G_{s,n}}{\partial z_j} = \frac{n}{r+2} G_{s+r-j+2,n} - \sum_{i=0}^{r} z_i \frac{\partial G_{s+r-j+2,n}}{\partial z_i}.$$

Proof: Straightforward.

For $n \ge t \ge k$ consider the following sequence:

$$B^t_{k,r} = \{\overbrace{0, \ldots, 0}^{n-t}, G_{t,t}, \ldots, G_{k,t}\}.$$

PROPOSITION 1.8: *$B^t_{k,r}$ is a Toeplitz sequence.*

Proof:

(i) $$\frac{\partial G_{s+1,t}}{\partial z_{j+1}} = \frac{\partial G_{s,t}}{\partial z_j} \quad \text{for } j < r.$$

This follows immediately from Lemma 1.7.

(ii) $$\frac{\partial G_{s,t}}{\partial z_r} = \frac{t}{r+2} G_{s+2,t} - \sum_{j=0}^{r} z_j \frac{\partial G_{s+2,t}}{\partial z_j}$$

by Lemma 1.7.

On the other hand, it follows from Lemma 1.6 that

$$\sum_{j=0}^{r} (r+2-j) z_j \frac{\partial G_{s+2,t}}{\partial z_j} = (t-s-2) G_{s+2,t}.$$

This is equivalent to

$$\sum_{j=0}^{r} z_j \frac{\partial G_{s+2,t}}{\partial z_j} = \frac{t-s-2}{r+2} G_{s+2,t} + \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial G_{s+2,t}}{\partial z_j}.$$

Therefore

$$\begin{aligned} \frac{\partial G_{s,t}}{\partial z_r} &= \frac{t}{r+2} G_{s+2,t} - \frac{t-s-2}{r+2} G_{s+2,t} - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial G_{s+2,t}}{\partial z_j} \\ &= \frac{s+2}{r+2} G_{s+2,t} - \sum_{j=0}^{r} \frac{j}{r+2} z_j \frac{\partial G_{s+2,t}}{\partial z_j}. \end{aligned}$$

This concludes the proof. ∎

So for given n and k, we can construct $n-k+1$ Toeplitz sequences: $B^n_{k,r}, B^{n-1}_{k,r}, \ldots, B^k_{k,r}$. The height of each one does not exceed n and they are linearly independent over K (see *Remark*). These sequences, therefore, form a basis for $T^n_{k,r}$.æ

2. Reduced Jacobian pairs

For a pair of polynomials $A, B \in K[x,y]$ let $[A,B]$ denote the Jacobian determinant $A_x B_y - A_y B_x$. The operation $[\ ,\ ]$ imposes a Lie-algebra structure on $K[x,y]$.

A pair $P, Q \in K[x,y]$ is called a **Jacobian pair** if $[P,Q] = 1$.

Definition 2.1: A Jacobian pair $P, Q \in K[x,y]$ is called a **reduced Jacobian pair** if:

$$P = x^n + P_{n-1}(y)x^{n-1} + \cdots + P_0(y),$$

$$Q = x^m + Q_{m-2}(y)x^{m-2} + \cdots + Q_0(y) \quad \text{and} \quad m \geq 2.$$

It is easy to see that every Jacobian pair P, Q with $\deg Q \geq 2$ can be brought to this form with the help of several linear and triangular automorphisms of $K[x,y]$. So in our treatment of the Jacobian Conjecture, we can concentrate our attention on reduced Jacobian pairs.

Let $P, Q \in K[x,y]$ be a reduced Jacobian pair. It will be convenient to consider P as a formal power series in x, including the negative powers:

$$P = x^n + P_{n-1}(y)x^{n-1} + \cdots + P_0(y) + P_{-1}(y)x^{-1} + \cdots.$$

We should, of course, keep in mind that $P_i(y) = 0$ for $i < 0$. Q, on the other hand, will be written in the standard polynomial form:

$$Q = x^m + Q_{m-2}(y)x^{m-2} + \cdots + Q_0(y).$$

THEOREM 2.2: *There exist $F_n, \ldots, F_{1-m} \in K[z_{m-2}, \ldots, z_0]$ such that:*

(i) $F = \{F_n, \ldots, F_{1-m}\} \in T^n_{1-m,m-2}$,

(ii) $P_i(y) = F_i(Q_{m-2}(y), \ldots, Q_0(y))$, $2-m \leq i \leq n$,

(iii) $F_{1-m}(Q_{m-2}(y), \ldots, Q_0(y)) = y/m$.

Proof: First we will prove the following statement: For every $s, 2-m \leq s \leq n$, there exists a Toeplitz sequence $F = \{F_n, \ldots, F_s\} \in T^n_{s,m-2}$ such that $P_i(y) = F_i(Q_{m-2}(y), \ldots, Q_0(y))$ for $s \leq i \leq n$.

We will prove this statement by induction on $n-s$. Note that $P_n(y) = 1$ and $\{1\} \in T^n_{n,m-2}$. So our statement is true for $s = n$. Now assume that it is true for some $s+1$. We will prove that it is then true for s, if $s \geq 2-m$. Consider the equation $[P,Q] = 1$. Since, by our convention, P is a power series in x and Q is a polynomial, we can expand $[P,Q]$ into a power series in x:

$$[P,Q] = \sum_{d=-\infty}^{n+m-2} A_d(y)x^d,$$

where $A_d(y)$ are some expressions containing the coefficients $P_i(y), Q_j(y)$ and their derivatives. It is easy to verify that

$$A_d(y) = -mP'_{d-m+1}(y) + \sum_{j=2}^{m} \{(d-m+1+j)P_{d-m+1+j}(y)Q'_{m-j}(y) - \\ - (m-j)\, P'_{d-m+1+j}(y)Q_{m-j}(y)\}.$$

Since $[P,Q] = 1$, we know that $A_d(y) = 0$ for $d > 0$ and $A_0(y) = 1$. Take $d = m+s-1$; $d > 0$ since $s \geq 2-m$. We then obtain the following equation:

$$mP'_s(y) = \sum_{j=2}^{m}(s+j)P_{s+j}(y)Q'_{m-j}(y) - (m-j)P'_{s+j}(y)Q_{m-j}(y). \tag{1}$$

Here we assume, of course, that $P_{s+j}(y) = 0$ for $s+j > n$.

We will integrate this equation, using our induction hypothesis. By this hypothesis there exists a Toeplitz sequence

$$\{F_n, \ldots, F_{s+1}\} \in T^n_{s+1,m-2}$$

such that $P_{s+j}(y) = F_{s+j}(Q_{m-2}(y), \ldots, Q_0(y))$ for $j \geq 1$. Thus

$$P'_{s+j}(y) = \sum_{k=2}^{m} \frac{\partial F_{s+j}}{\partial z_{m-k}}(Q_{m-2}(y), \ldots, Q_0(y))Q'_{m-k}(y)$$

and equation (1) can be rewritten as follows:

$$\begin{aligned} mP'_s &= \sum_{j=2}^{m}(s+j)F_{s+j}(Q_{m-2}, \ldots, Q_0)Q'_{m-j} \\ &\quad - \sum_{j=2}^{m}(m-j)Q_{m-j}\sum_{k=2}^{m}\frac{\partial F_{s+j}}{\partial z_{m-k}}(Q_{m-2}, \ldots, Q_0)Q'_{m-k}. \end{aligned} \tag{2}$$

Replacing j by k in the first sum and changing the order of summation in the second sum, we obtain

$$\begin{aligned} P'_s = \sum_{k=2}^{m} \Bigg\{ &\frac{s+k}{m}F_{s+k}(Q_{m-2}, \ldots, Q_0) \\ &- \sum_{j=2}^{m}\frac{m-j}{m}Q_{m-j}\frac{\partial F_{s+j}}{\partial z_{m-k}}(Q_{m-2}, \ldots, Q_0)\Bigg\} Q'_{m-k}. \end{aligned} \tag{3}$$

Since the functions F_{s+j} are terms of a Toeplitz sequence, it follows from Definition 1.1 that

$$\frac{\partial F_{s+j}}{\partial z_{m-k}} = \frac{\partial F_{s+k}}{\partial z_{m-j}}.$$

Therefore

$$\begin{aligned} P_s' = \sum_{k=2}^{m} \Bigg\{ & \frac{s+k}{m} F_{s+k}(Q_{m-2}, \dots, Q_0) \\ & - \sum_{j=2}^{m} \frac{m-j}{m} Q_{m-j} \frac{\partial F_{s+k}}{\partial z_{m-j}}(Q_{m-2}, \dots, Q_0) \Bigg\} Q'_{m-k}. \end{aligned} \tag{4}$$

By Proposition 1.4 there exists $F_s(z_{m-2}, \dots, z_0)$ such that $\{F_n, \dots, F_{s+1}, F_s\}$ is a Toeplitz sequence. Using again Definition 1.1 we obtain

$$\frac{s+k}{m} F_{s+k} - \sum_{j=2}^{m} \frac{m-j}{m} z_{m-j} \frac{\partial F_{s+k}}{\partial z_{m-j}} = \frac{\partial F_{s+k-2}}{\partial z_{m-2}} = \frac{\partial F_s}{\partial z_{m-k}}.$$

Therefore

$$P_s' = \sum_{k=2}^{m} \frac{\partial F_s}{\partial z_{m-k}}(Q_{m-2}, \dots, Q_0) Q'_{m-k}.$$

In other words

$$P_s'(y) = \frac{dF_s(Q_{m-2}(y)), \dots, Q_0(y))}{dy}.$$

Since F_s is defined up to an additive constant (Lemma 1.3), we obtain that $P_s(y) = F_s(Q_{m-2}(y), \dots, Q_0(y))$ and the induction step is concluded.

So now we have a Toeplitz sequence $\{F_n, \dots, F_{2-m}\}$ such that

$$P_i(y) = F_i(Q_{m-2}(y), \dots, Q_0(y)) \quad \text{for } 2-m \le i \le n.$$

Note that $F_i(Q_{m-2}(y), \dots, Q_0(y)) = 0$ for $2-m \le i < 0$. If we apply the same method to $s = 1-m$, we obtain an analog of equation (1):

$$1 + mP'_{1-m} = \sum_{j=2}^{m} (1-m+j) P_{1-m+j} Q'_{m-j} - (m-j) P'_{1-m+j} Q_{m-j}. \tag{5}$$

Integrating equation (5) in the same way as we did before, we obtain

$$\frac{1}{m} + P'_{1-m}(y) = \frac{dF_{1-m}(Q_{m-2}(y), \dots, Q_0(y))}{dy} \tag{6}$$

or

$$\frac{y}{m} + P_{1-m}(y) = F_{1-m}(Q_{m-2}(y), \ldots, Q_0(y)). \tag{7}$$

But $P_{1-m}(y) = 0$ since $m \geq 2$. Therefore

$$F_{1-m}(Q_{m-2}(y), \ldots, Q_0(y)) = \frac{y}{m}.$$

This concludes the proof. ∎

Theorem 2.2 supplies necessary conditions for P, Q to be a reduced Jacobian pair. In fact, these conditions are sufficient as well.

THEOREM 2.3: *Set $P = x^n + P_{n-1}(y)x^{n-1} + \cdots + P_0(y)$ and set*

$$Q = x^m + Q_{m-2}(y)x^{m-2} + \cdots + Q_0(y), \quad m \geq 2.$$

Then the following are necessary and sufficient conditions for P, Q to be a Jacobian pair:
There exists a Toeplitz sequence $\{F_n, \ldots, F_{1-m}\}$ with the following properties:
(i) $P_i(y) = F_i(Q_{m-2}(y), \ldots, Q_0(y))$ for $0 \leq i \leq n$,
(ii) $F_i(Q_{m-2}(y), \ldots, Q_0(y)) = 0$ for $2 - m \leq i < 0$,
(iii) $F_{1-m}(Q_{m-2}(y), \ldots, Q_0(y)) = \frac{y}{m}$.

Proof: The necessary part follows from Theorem 2.2. To prove that the conditions are sufficient, we simply reverse the proof of Theorem 2.2 (all steps in this proof are reversible). This concludes the proof. ∎

Recall now the basis $B^n_{k,r}, \ldots, B^t_{k,r}, \ldots, B^k_{k,r}$ for $T^n_{k,r}$ (this basis was constructed in Section 1).

$$B^t_{k,r} = \{\overbrace{0, \ldots, 0}^{n-t}, G_{t,t}, \ldots, G_{k,t}\}.$$

Our Toeplitz sequence $F = \{F_n, \ldots, F_{1-m}\}$ can be written as a linear combination of the elements of this basis: $F = \sum_{t=1-m}^{n} c_t B^t_{1-m,m-2}$ or, in other words,

$$F_i = \sum_{t=i}^{n} c_t G_{i,t}, \quad 1 - m \leq i \leq n, \quad c_n = 1.$$

Thus, we obtain $m+n-1$ equations with $m+n-1$ indeterminate constants:

$$P_i(y) = \sum_{t=i}^{n} c_t G_{i,t}(Q_{m-2}(y), \ldots, Q_0(y)), \quad 0 \le i \le n-1,$$

$$0 = \sum_{t=i}^{n} c_t G_{i,t}(Q_{m-2}(y), \ldots, Q_0(y)), \quad 2-m \le i < 0,$$

$$\frac{y}{m} = \sum_{t=1-m}^{n} c_t G_{1-m,t}(Q_{m-2}(y), \ldots, Q_0(y)).$$

If we consider only the last $m-1$ equations, we obtain a system of $m-1$ equations with $m+n-1$ indeterminate constants:

$$\begin{aligned} &\sum_{t=i}^{n} c_t G_{i,t}(Q_{m-2}(y), \ldots, Q_0(y)) = 0, \quad 2-m \le i < 0, \\ &\sum_{t=1-m}^{n} c_t G_{1-m,t}(Q_{m-2}(y), \ldots, Q_0(y)) = \frac{y}{m}. \end{aligned} \tag{8}$$

PROPOSITION 2.4: *Set $Q = x^m + Q_{m-2}(y)x^{m-2} + \cdots + Q_0(y), m \ge 2$. The polynomial Q has a monic in x Jacobian mate of x-degree n iff there exist constants $c_{1-m}, \ldots, c_n$ satisfying the system* (8).

Proof:

(i) If such a Jacobian mate exists, then the existence of the constants follows from Theorem 2.3.

(ii) Assume that $c_{1-m}, \ldots, c_n$ satisfy system (8). Then we can define $F_i = \sum_{t=i}^{n} c_t G_{i,t}$ for $i = 1-m, \ldots, n$. These functions form a sequence $F = \{F_n, \ldots, F_{1-m}\}$. The sequence F is, obviously, a Toeplitz sequence since $F = \sum_{t=1-m}^{n} c_t B^t_{1-m,m-2}$ and $\{B^t_{1-m,m-2}\}$ form a basis in $T^n_{1-m,m-2}$.

Set now $P_i(y) = F_i(Q_{m-2}(y), \ldots, Q_0(y))$ for $0 \le i \le n$ and set $P = x^n + P_{n-1}(y)x^{n-1} + \cdots + P_0(y)$. Then $[P, Q] = 1$ by Theorem 2.3. This concludes the proof. ∎

Remark: There is another way of looking at the polynomials $G_{i,t}(z_r, \ldots, z_0)$ (see Section 1). Consider the following polynomial:

$$G(x, z_r, \ldots, z_0) = x^{r+2} + z_r x^r + \cdots + z_0.$$

It is convenient to rewrite it as

$$G(x, z_r, \ldots, z_0) = x^{r+2}(1 + z_r x^{-2} + \cdots + z_0 x^{-r-2}).$$

Now we can construct some fractional powers of $G(x, z_r, \ldots, z_0)$ as formal power series in x including negative powers of x:

$$G^{t/(r+2)} = \sum_{i=-\infty}^{t} G_{i,t}(z_r, \ldots, z_0)x^i. \tag{9}$$

It is easy to see that the coefficients $G_{i,t}$ appearing in this expansion are identical with the polynomials $G_{i,t}$ defined in Section 1. Let $< G^{t/(r+2)} >_k$ denote the finite portion of the expansion (9) starting from k:

$$< G^{t/(r+2)} >_k = \sum_{i=k}^{t} G_{i,t}(z_r, \ldots, z_0)x^i. \tag{10}$$

We will call expressions of this form truncated fractional powers of G.

Returning to Proposition 2.4 we now see that

$$\begin{aligned} P_i(y) &= \sum_{t=i}^{n} c_t G_{i,t}(Q_{m-2}(y), \ldots, Q_0(y)) \quad \text{and} \\ P(x,y) &= \sum_{i=0}^{n} P_i(y)x^i = \sum_{i=0}^{n}\sum_{t=i}^{n} c_t G_{i,t}(Q_{m-2}(y), \ldots, Q_0(y))x^i \\ &= \sum_{t=0}^{n} c_t \sum_{i=0}^{t} G_{i,t}(Q_{m-2}(y), \ldots, Q_0(y))x^i = \sum_{t=0}^{n} c_t < Q(x,y)^{t/m} >_0 . \end{aligned}$$

Thus P is a linear combination of truncated fractional powers of Q. If we consider the whole system of equations appearing in Proposition 2.4, we obtain the following:

$$P(x,y) + \frac{y}{m}x^{1-m} = \sum_{t=1-m}^{n} c_t < Q(x,y)^{t/m} >_{1-m} .$$

3. The fundamental system for P, Q

Recalling Theorem 2.3, we see that in order for P, Q to be a reduced Jacobian pair, the coefficients $Q_i(y)$ of Q must satisfy a system of polynomial equations:

$$\begin{gathered} F_{-1}(Q_{m-2}(y), \ldots, Q_0(y)) = 0, \\ \vdots \\ F_{2-m}(Q_{m-2}(y), \ldots, Q_0(y)) = 0, \\ F_{1-m}(Q_{m-2}(y), \ldots, Q_0(y)) = y/m, \end{gathered}$$

where $F_i(z_{m-2}, \ldots, z_0)$ are terms of a certain Toeplitz sequence. This system will be called the fundamental system for the pair P, Q.

The fundamental system has some very interesting properties. Let $S(z_{m-2}, \ldots, z_0)$ denote the Jacobian matrix

$$\left[\frac{\partial F_i}{\partial z_j}\right], \quad 1-m \leq i \leq -1, \quad 0 \leq j \leq m-2.$$

Set $\tilde{S}(y) = S(Q_{m-2}(y), \ldots, Q_0(y))$ and set $\Delta = \det\tilde{S}(y)$. (Both S and $\tilde{S}$ are square $m-1 \times m-1$ matrices.)

Let $V(y)$ denote the column vector of derivatives:

$$V(y) = \begin{pmatrix} Q'_{m-2}(y) \\ \vdots \\ Q'_0(y) \end{pmatrix}.$$

Differentiating the system (11) with respect to y, we obtain the following matrix equation:

$$\tilde{S}(y)V(y) = \begin{pmatrix} 0 \\ \vdots \\ \frac{1}{m} \end{pmatrix}. \tag{12}$$

The immediate consequence of this is:

Lemma 3.1: *The derivatives $Q'_{m-2}(y), \ldots, Q'_0(y)$ do not vanish simultaneously.*

Proof: Obvious. ∎

It is easy to see that both $S(z_{m-2}, \ldots, z_0)$ and $\tilde{S}(y)$ are Toeplitz matrices. It follows immediately from the definition of a Toeplitz sequence. The fact itself was the reason for the name "Toeplitz sequence". We will use this fact now.

Theorem 3.2: *$\tilde{S}(y)$ is an invertible matrix, the entries of $\tilde{S}^{-1}(y)$ are polynomials in y and $\Delta = \text{const} \neq 0$.*

Proof: Let A_i denote the following row vector:

$$\left[\frac{\partial F_i}{\partial z_{m-2}}(Q_{m-2}(y), \ldots, Q_0(y)), \ldots, \frac{\partial F_i}{\partial z_0}(Q_{m-2}(y), \ldots, Q_0(y))\right].$$

We have from Definition 1.1:

(i) $\frac{\partial F_{i+1}}{\partial z_{j+1}}(Q_{m-2}(y),\dots,Q_0(y)) = \frac{\partial F_i}{\partial z_j}(Q_{m-2}(y),\dots,Q_0(y)),\ 0 \le j < m-2,$

(ii) $$\frac{\partial F_i}{\partial z_{m-2}}(Q_{m-2}(y),\dots,Q_0(y)) = \frac{i+2}{m}F_{i+2}(Q_{m-2}(y),\dots,Q_0(y)) - \sum_{j=1}^{m-2}\frac{j}{m}Q_j(y)\frac{\partial F_{i+2}}{\partial z_j}(Q_{m-2}(y),\cdots,Q_0(y)).$$

If $i=-2$, then $\frac{i+2}{m}=0$, and if $1-m \le i < -2$, then $F_{i+2}(Q_{m-2}(y),\dots,Q_0(y)) = 0$ according to system (11). In any case:

(iii) $$\begin{aligned}\frac{\partial F_i}{\partial z_{m-2}}&(Q_{m-2}(y),\dots,Q_0(y))\\ &= -\sum_{j=0}^{m-2}\frac{j}{m}Q_j(y)\frac{\partial F_{i+2}}{\partial z_j}(Q_{m-2}(y),\dots,Q_0(y))\\ &= -\sum_{j=1}^{m-2}\frac{j}{m}Q_j(y)\frac{\partial F_{i+1}}{\partial z_{j-1}}(Q_{m-2}(y),\dots,Q_0(y)),\ 1-m \le i < -1.\end{aligned}$$

Set

$$T = \begin{pmatrix} 0 & & 1 & 0 & & \dots & 0\\ -\frac{m-2}{m} & Q_{m-2} & 0 & 1 & & & 0\\ \vdots & & \vdots & & \ddots & & \\ & & & & & & 1\\ -\frac{1}{m} & Q_1 & & 0\dots & \dots & & 0 \end{pmatrix}$$

Then we obtain from (i) and (iii) that

$$A_i = A_{i+1}T = \cdots = A_{-1}T^{-1-i}, \quad 1-m \le i \le -1.$$

Consider the column vector

$$V = \begin{pmatrix} Q'_{m-2}\\ \vdots\\ Q'_0 \end{pmatrix}.$$

It follows from (12) that

$$A_{-1}T^{-1-i}V = 0, \quad \text{for } 1-m < i \le -1,$$

$$A_{-1}T^{m-2}V = \frac{1}{m}.$$

Let B_k denote the column vector T^kV, $0 \le k \le m-2$. Set $B = (B_0, \dots, B_{m-2})$ — the $m-1 \times m-1$ matrix, whose columns are vectors B_k. Then it is easy to see that

$$\tilde{S}B = \begin{pmatrix} & & & \frac{1}{m} \\ 0 & & \frac{1}{m} & * \\ & \cdot^{\cdot^{\cdot}} & * & * \\ \frac{1}{m} & & * & * \end{pmatrix}.$$

Therefore $\Delta \cdot \det B = \left[-\frac{1}{m}\right]^{m-1}$ and Δ and $\det B$ are non-zero constants. This concludes the proof. ∎

4. Some partial cases

The last equation of the fundamental system (11) implies that $K[Q_{m-2}(y), \dots, Q_0(y)] = K[y]$. Thus, we have an embedding of a line into K^{m-1}. We will use this fact in the proof of the following result (cases (i) and (iii)):

THEOREM 4.1: *Let $Q = x^m + Q_k(y)x^k + Q_r(y)x^r, m > k > r$. If Q has a Jacobian mate $P = x^n + P_{n-1}(y)x^{n-1} + \dots + P_0(y)$, then $K[P,Q] = K[x,y]$.*

Proof: We will consider several cases:

(i) $m = 2$

In this case the result is very simple and well-known. We will give a proof for the sake of completeness. If $m = 2$, then $Q = x^2 + Q_1(y)x + Q_0(y)$. Using the automorphism $x \to x - \frac{Q_1(y)}{2}$, $y \to y, Q$ is reduced to the form $x^2 + \tilde{Q}_0(y)$ and the last equation of the fundamental system (11) is $F_{-1}(\tilde{Q}_0(y)) = \frac{y}{2}$. Therefore $\tilde{Q}_0(y) = \alpha y + \beta, \alpha \in K^*, \beta \in K$. The rest is obvious.

(ii) $m > 2,\ r = 0,\ k > 1$

$$Q = x^m + Q_k(y)x^k + Q_0(y)$$

In this case $Q_0' = c_0 \in K^*$. Indeed, assume that $Q_0'(\alpha) = 0$ for some $\alpha \in K$. Then $Q_x(0,\alpha) = Q_y(0,\alpha) = 0$, which is impossible since Q has a Jacobian mate. So $Q_0 = c_0y + \beta, Q_x = x^{k-1}(mx^{m-k} + kQ_k), Q_y = Q_k'x^k + c_0$. Then the polynomials $mx^{m-k} + kQ_k$ and $Q_k'x^k + c_0$ cannot have common zeroes, which easily implies that

$$\operatorname{Res}_x(mx^{m-k} + kQ_k, Q_k'x^k + c_0) = \lambda \in K^*.$$

This resultant is $m \times m$ and has the following form:

$$\begin{vmatrix} m & 0\dots0 & & kQ_k & 0 & & \dots0 \\ 0 & m\,0\dots & & 0 & kQ_k & 0 & \dots0 \\ \vdots & & & & & \ddots & \\ 0 & \dots0 & m & 0\dots\dots\dots & & 0 & kQ_k \\ Q'_k & 0\dots & 0 & c_0 & & & \\ & & \vdots & & & & \\ 0 & \dots0 & Q'_k & 0\dots\dots\dots & & 0 & c_0 \end{vmatrix}$$

If $Q'_k \neq 0$, then its leading term is, obviously, given by $(kQ_k)^k(Q'_k)^{m-k}$. Therefore $k\deg Q_k + (m-k)(\deg Q_k - 1) = 0$ or $m\deg Q_k - m + k = 0$, which is impossible since $k > 0$. Thus $Q'_k = 0, Q_k = c_k \in K$ and $Q = x^m + c_k x^k + c_0 y + \beta$. The rest easily follows.

(iii) $m > 2$, $r = 0$, $k = 1$

$$Q = x^m + Q_1(y)x + Q_0(y), \quad Q_x = mx^{m-1} + Q_1, \quad Q_y = Q'_1 x + Q'_0.$$

Then, as previously, $\mathrm{Res}_x(Q_x, Q_y) = \lambda \in K^*$.

The resultant has the following form:

$$\begin{vmatrix} m & 0 & \dots & 0 & & Q_1 \\ Q'_1 & Q'_0 & 0 & \dots & & 0 \\ 0 & Q'_1 & Q'_0 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0\dots & \dots & \dots & 0 & Q'_1 & Q'_0 \end{vmatrix}$$

Thus

$$\lambda = mQ'^{m-1}_0 + (-1)^{m-1}Q_1 Q'^{m-1}_1. \tag{13}$$

Let $d_0 = \deg Q_0, d_1 = \deg Q_1$. Assume first that $d_1 \neq 0, d_0 \neq 0$. The last equation of (11) implies that the map $y \to (Q_1(y), Q_0(y))$ is an embedding of K^1 into K^2. Then, by the Abhyankar–Moh Theorem (see [1]), either d_1 divides d_0 or d_0 divides d_1. On the other hand, from (13), $(m-1)(d_0-1) = d_1 + (m-1)(d_1-1)$ or $(m-1)d_0 = md_1$, which is impossible since $m > 2$. Therefore either $d_1 = 0$ or $d_0 = 0$. If $d_0 = 0$, then $d_1 + (m-1)(d_1-1) = 0$, which is impossible. Thus

$d_1 = 0, Q_1' = 0$ and $d_0 = 1$. So $Q_1 = c_1 \in K$ and $Q_0 = c_0 y + \beta, c_0 \in K^*, \beta \in K$. Hence $Q = x^m + c_1 x + c_0 y + \beta$ and the rest easily follows.

(iv) $m > k > r > 0$

If $r > 1$, then $Q_x(0,y) = Q_y(0,y) = 0$. Therefore $r = 1, Q = x^m + Q_k x^k + Q_1 x$.

$$Q_x = mx^{m-1} + kQ_k x^{k-1} + Q_1, \quad Q_y = Q_k' x^k + Q_1' x.$$

If there exists $\alpha \in K$ such that $Q_1(\alpha) = 0$, then $Q_x(0,\alpha) = Q_y(0,\alpha) = 0$, which is impossible. Thus $Q_1 = c_1 \in K$,

$$Q = x(x^{m-1} + Q_k x^{k-1} + c_1), \quad Q_x = mx^{m-1} + kQ_k x^{k-1} + c_1, \quad Q_y = Q_k' x^k.$$

If there exists $\alpha \in K$ such that $Q_k'(\alpha) = 0$, then $Q_y(x,\alpha) = 0$ for every x. Hence $Q_x(x,\alpha)$ cannot have zeroes, which is impossible since $m > 2$. Therefore $Q_k = c_k y + \beta, c_k \in K^*$, $\beta \in K$. Hence $Q = x^m + (c_k y + \beta)x^k + c_1 x$. Consider the following automorphism τ of $K[x,y]$:

$$\tau: \quad x \to c_k x, \quad y \to \frac{1}{c_k}(y - (c_k x)^{m-k} - \beta).$$

It is easy to see that $\tau(Q) = c_k^k y x^k + c_1 c_k x$ and it is very easy to prove by standard methods that $\tau(Q)$ cannot have a Jacobian mate. This concludes the proof. ∎

5. Some conjectures and observations

What properties of Q were used in the proof of Theorem 4.1? In fact we used only two facts:

1. $K[Q_k(y), Q_r(y)] = K[y]$,
2. $\text{Res}_x(Q_x, Q_y) = \lambda \in K^*$.

It is a well-known fact (see [4] for example) that if Q is a member of a Jacobian pair, then $\text{res}_x(Q_x, Q_y) =$ non-zero constant. In the case $m = 3$ there is a striking similarity between this resultant and $\det B = -(Q_0'^2 + \frac{1}{3} Q_1 Q_1'^2)$ which is also a non-zero constant. We would like to state the following:

CONJECTURE: $\text{Res}_x(Q_x, Q_y) = \alpha \det B, \ \alpha \neq 0$.

Suppose now that $\text{Res}_x(Q_x, Q_y)$ is a non-zero constant and

$$K[Q_{m-2}(y), \ldots, Q_0(y)] = K[y].$$

These are the only properties of Q used in the proof of case (iii) of Theorem 4.1. Are these properties sufficient for the general case? The answer to this question is no, as can be easily seen from the following counter-example:

$$Q = x^m + yx^k + x, \quad m > 3, \quad m - 1 > k > 1.$$

The Abhyankar–Moh Theorem played a very important role in the proof of Theorem 4.1. There exists a well-known generalization of the Abhyankar–Moh Theorem for embeddings of a line into K^r for $r > 3$ (see [5]). Namely: If $K[A_1(y), \ldots, A_r(y)] = K[y]$, then there exists an automorphism τ of $K[z_1, \ldots, z_r]$ such that $\tau(z_i) = P_i(z_1, \ldots, z_r)$, $1 \leq i \leq r$, and $P_i(0, \ldots, 0, y) = A_i(y)$. Unfortunately, there are no simple relations between the degrees of A_i's if $r > 3$. So, the question arises of how to use this generalization of the Abhyankar–Moh Theorem for the proof of the general case of the Jacobian Conjecture for $K[x, y]$ in a manner similar to Theorem 4.1.

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